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# The quasitopos hull of the category of uniform spaces - a correction

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*Abstract*

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We present a description of the quasitopos hull of the category of uniform spaces. This corrects our earlier description published in *Topology Appl.* 27.

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In our paper [4] we attempted to describe the quasitopos hull of the category **Unif** of uniform spaces. Unfortunately, that description was incorrect, and the reason was a mistake in [2] where it was claimed that, given a topological category $\mathscr{K}$ which is a full, finally dense subcategory of a quasitopos $\mathscr{Q}$ then the initial lifts of all $[K_1, K_2]^{\#}$ with $K_1, K_2 \in \mathscr{K}$ form the quasitopos hull of $\mathscr{K}$. (We denote power objects by $[A, B]$ and objects representing strong partial morphisms with codomain $A$ by $A^{\#}$.) The statement has been corrected in [5] as follows:

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**Theorem.** *Let $\mathcal{K}$ be a topological category which is a full, finally dense subcategory of a concrete quasitopos $\mathcal{Q}$. Then $\mathcal{Q}$ is a quasitopos hull of $\mathcal{K}$ iff all objects $[K_1, K_2^\#]$ with $K_1, K_2 \in \mathcal{K}$ are initially dense in $\mathcal{Q}$.*

**Corollary.** *Let $\mathcal{K}$ be a topological category which is a full, finally dense subcategory of a concrete quasitopos $\mathcal{Q}$. Then the initial hull of all objects $[K_1, K_2^\#]$ of $\mathcal{Q}$ with $K_1 \in \mathcal{Q}$ and $K_2 \in \mathcal{K}$ forms a quasitopos hull of $\mathcal{K}$.*

In fact, from the proof of the above theorem it follows readily that the quasitopos hull of $\mathcal{K}$ is the initial hull of all objects $[K_1, K_2^\#]$ with $K_1, K_2 \in \mathcal{K}$. For $K_1 \in \mathcal{Q}-\mathcal{K}$ use a final sink $(A_i \xrightarrow{\alpha_i} K_1)_{i \in I}$ with $A_i \in \mathcal{K}$: then $([K_1, K_2^\#] \xrightarrow{[\alpha_i, 1]} [A_i, K_2^\#])_{i \in I}$ is an initial source, thus, $[K_1, K_2^\#]$ lies in the quasitopos hull.

In what follows we work with *semi-uniform spaces*, see [11], i.e., pairs $(X, \alpha)$ where $\alpha$ is a filter (w.r.t. the ordering by inclusion) of symmetric vicinities of the diagonal in $X \times X$ (i.e., of sets $U \subseteq X \times X$ with $\Delta_X \subseteq U = U^{-1}$). Let **SUnif** be the category of semi-uniform spaces, where morphisms $f: (X, \alpha) \to (Y, \beta)$ are functions $f: X \to Y$ such that $U \in \beta$ implies $(f \times f)^{-1}(U) \in \alpha$.

**Unif** is, obviously, a full subcategory of **SUnif**. It is finally dense in **SUnif** because, as proved in [7], every semi-uniform space is a quotient of a uniform space. It is easy to verify that both quotients and finite coproducts are stable under pullbacks in **SUnif**. Thus, by [4, Remark 8], we have the concrete quasitopos $A^*$**SUnif** of *filters of semi-uniformities*: objects are pairs $(X, \mathcal{A})$ where $X$ is a set and $\mathcal{A}$ is a filter of semi-uniform spaces $(Y, \beta)$, $Y \subseteq X$, under the ordering $<$ defined by $(Y, \beta) < (Y', \beta')$ iff $Y \subseteq Y'$ and the inclusion is a morphism $(Y, \beta) \to (Y', \beta')$ of **SUnif**. *Filter* means that

(a) $(Y, \beta) \in \mathcal{A}$ implies $(Y', \beta') \in \mathcal{A}$ whenever $(Y', \beta') < (Y, \beta)$,

(b) for $(Y', \beta')$, $(Y'', \beta'') \in \mathcal{A}$ there exists $(Y, \beta) \in \mathcal{A}$ with $(Y', \beta') < (Y, \beta)$ and $(Y'', \beta'') < (Y, \beta)$,

(c) for each $x \in X$ there exists $(Y, \beta) \in \mathcal{A}$ with $x \in Y$.

Morphisms $f: (X, \mathcal{A}) \to (X', \mathcal{A}')$ are functions $f: X \to X'$ such that for each $(Y, \beta) \in \mathcal{A}$ there exists $(Y', \beta') \in \mathcal{A}'$ with $f(Y) \subseteq Y'$ and such that the domain-range restriction of $f$ defines a morphism $(Y, \beta) \to (Y', \beta')$ of **SUnif**.

We consider **SUnif** as a full, concrete subcategory of $A^*$**SUnif** by identifying each $(X, \alpha) \in$ **SUnif** with $(X, \{(Y, \beta): (Y, \beta) < (X, \alpha)\})$.

**Definition.** (1) A semi-uniformity $\alpha$ is called *simple* if it has a base formed by a single vicinity $U$, i.e., $\alpha$ consists of all symmetric supersets of $U$. The semi-uniformity is then denoted by $[U]$.

(2) A filter of semi-uniformities is called *simple* if it has a basis of simple semi-uniformities. That is, $(X, \mathcal{A})$ is simple if for each $(Y, \beta) \in \mathcal{A}$ there is $(Y', [U]) \in \mathcal{A}$ with $(Y, \beta) < (Y', [U])$.

(3) A filter of semi-uniformities is called *saturated* if it is an intersection of simple filters.

(4) The full subcategory of $A^*$**SUnif** of all saturated filters is denoted by **SF**.

**Lemma 1.** *The power object of semi-uniform spaces $A=(X, \alpha)$, $B=(Y, \beta)$ in the category $A^*$**SUnif** is the following saturated filter of semi-uniformities:*

$$[A, B]=(\hom(A, B), \mathscr{C})$$

*where $\mathscr{C}$ consists of all semi-uniform spaces $(M, \mu)$ satisfying the following conditions:*

(1) *$M \subseteq \hom(A, B)$ is an equicontinuous set, i.e., for each $V \in \beta$ there exists $U \in \alpha$ with $\bigcup_{f \in M} f \times f(U) \subseteq V$; in other words, there is a function $r: \beta \to \alpha$ coherent in the sense that $\bigcup_{f \in M} f \times f(rV) \subseteq V$ for all $V \in \beta$.*

(2) *$\mu$ contains, for some coherent function $r: \beta \to \alpha$, all the vicinities $V^{(r)} = \{(f, g) \in M \times M: f \times g(rV) \subseteq V\}$ for $V \in \beta$, i.e., $\{V^{(r)}: V \in \beta\} \subseteq \mu$.*

**Proof.** (I) Evaluation is a morphism $\mathrm{ev}: [A, B] \times A \to B$ in $A^*$**SUnif.** To prove this, it clearly suffices to show, for each equicontinuous set $M \subseteq \hom(A, B)$ and every coherent function $r$, that the restriction of the evaluation is a morphism $\mathrm{ev}: (M, \mu) \times A \to B$ of **SUnif,** where $\mu$ is the semi-uniformity with the subbasis $\{V^{(r)}: V \in \beta\}$. This is obvious: for each $V \in \beta$ we have vicinities $V^{(r)}$ and $V^* = rV \in \alpha$ such that given $(f, g) \in V^{(r)}$ and $(x, y) \in V^*$ then

$$(f(x), g(y)) \in (f \times g)(V^*) \subseteq V.$$

(II) Given an object $D$ in $A^*$**SUnif** and a morphism $f: D \times A \to B$, the function $\hat{f}: D \to [A, B]$, $d \mapsto f(d, -)$, is a morphism, too. We prove this first in case $D=(T, \delta)$ is a semi-uniform space:

(a) Each $f(d, -): A \to B$ is clearly uniformly continuous.

(b) $\hat{f}$ is a morphism, i.e., $T=\hat{f}(M)$ is an equicontinuous set and, for some coherent map $r$, the range restriction of $\hat{f}$ is uniformly continuous from $D$ to $(M, \mu)$. In fact, for each $V \in \beta$ there exist $sV \in \delta$ and $rV \in \alpha$ such that

$$(d_1, d_2) \in sV \text{ and } (x, y) \in rV \text{ imply } (f(d_1, x), f(d_2, y)) \in V.$$

It follows that the map $r: \beta \to \alpha$ is coherent: for each $f(d, -) \in M$ we have $(d, d) \in sV$ and thus $(f(d, -) \times f(d, -))(rV) \subseteq V$. The continuity of the restriction of $\hat{f}$ is obvious: for each $V^{(r)}$ where $V \in \beta$ we have $sV \in \delta$ for which $(d_1, d_2) \in sV$ implies $(\hat{f}(d_1), \hat{f}(d_2)) \in V^{(r)}$.

Next, let $D$ be a filter of semi-uniformities $D_i$, $i \in I$. Since each of the restrictions $f_i: D_i \times A \to B$ of $f: D \times A \to B$ is a morphism, it follows from the above that each $\hat{f}_i: D_i \to [A, B]$ is a morphism and thus $\hat{f}: D \to [A, B]$ is a morphism since the sink of inclusion maps $(D_i \hookrightarrow D)_I$ is final in $A^*$**SUnif**.

(III) Finally, let us verify that $[A, B]$ is a saturated filter. If $B$ is a simple semi-uniform space, then $[A, B]$ is, obviously, a simple filter of semi-uniformities. In general, each semi-uniform space $B$ has an initial source $(B \xrightarrow{\mathrm{id}} B_i)_I$ with each $B_i$ simple, and this yields an initial source $([A, B] \xrightarrow{[A, \mathrm{id}]} [A, B_i])_I$ whose codomains are simple filters—thus the domain is saturated. □

**Lemma 2.** *For each semi-uniform space K the object $K^{\#}$ of* $A^*$**SUnif** *is a semi-uniform space.*

**Proof.** It is easy to verify that if $K=(X, \alpha)$ then $K^{\#}=(X\cup\{\infty\}, \alpha^{\#})$ where $\alpha^{\#}=\{U\cup(\{\infty\}\times X)\cup(X\times\{\infty\})\cup\{\infty,\infty\}: U\in\alpha\}$. $\square$

**Characterization Theorem.** *The category* **SF** *of saturated filters of semi-uniformities is a quasitopos hull of* **Unif.**

**Proof.** **SUnif** is clearly closed under quotients in $A^*$**SUnif.** Since each semi-uniform space is a quotient of a uniform space (in **SUnif**), it follows that not only **SUnif** but even **Unif** is finally dense in $A^*$**SUnif.** By the above corollary, it is sufficient to prove that the initial hull of all power objects $[K_1, K_2^{\#}]$ where $K_2$ is a uniform space is equal to **SF**. By Lemmas 1 and 2, these power objects lie in **SF** for $K_1\in$ **SUnif**. For $K_1$ general choose a final sink $(L_i\to K_1)$ with $L_i\in$ **Unif**; we obtain an initial lift $([K_1, K_2^{\#}]\to[L_i, K_2^{\#}])$ in $A^*$**SUnif**. Since **SF** is clearly closed in $A^*$**SUnif** under initial lifts, it follows that $[K_1, K_2^{\#}]\in$ **SF**.

Thus, it is sufficient to prove that each simple filter $(X, \mathscr{A})$ of semi-uniformities is an initial lift of an object $[P, R]$ where $P$ is a semi-uniform space and $R=S^{\#}$ for some uniform space $S$. Let $R=\{0, 1\}^{\#}$ where $\{0, 1\}$ is the discrete uniform space, i.e., $R$ is the simple semi-uniform space on the set $\{0, 1, \infty\}$ whose generating vicinity is

$$W=\{0,1,\infty\}\times\{0,1,\infty\}-\{(0,1),(1,0)\}.$$

Let $\mathscr{A}_0$ be the basis of $\mathscr{A}$ formed by all simple semi-uniform spaces in $\mathscr{A}$, and for each $A\in\mathscr{A}_0$, denote by $V_A$ the generating vicinity. Denote further by $P$ the following semi-uniform space: its underlying set is $X\times X\times\mathscr{A}_0$ and the following vicinities $U_A$, $A\in\mathscr{A}_0$, form a basis of its semi-uniformity:

$$U_A=\{(r,s)\in P\times P: (x,y)\in V_A \Rightarrow ((e(x)(r), e(y)(s))\in W\}$$

where $e: X\to R^P$ is defined by

$$e(x)(r)=\begin{cases}0, & \text{if } r=(x,x,A) \text{ for some } A\in\mathscr{A}_0,\\ 1, & \text{if } r=(x,y,A) \text{ for some } A\in\mathscr{A}_0\\ & \quad\text{with } (x,y)\notin V_A \text{ and } y\in A,\\ \infty, & \text{else.}\end{cases}$$

(To verify that $\{U_A\}$ is a basis of vicinities observe that each $U_A$ contains the diagonal of $P$, and that $U_A\subseteq U_{A_1}\cap U_{A_2}$ holds whenever $V_{A_1}\cup V_{A_2}\subseteq V_A$.) We are going to prove that

$$e:(X,\mathscr{A})\to[P,R] \text{ is initial in } A^*\mathbf{SUnif}.$$

(a) For each $x\in X$, $e(x): P\to R$ is uniformly continuous since $e(x)\times e(x)(U_{\{x\}})\subseteq W$.

(b) $e$ is a morphism since for each $A \in \mathscr{A}_0$, $M = e(A)$ is equicontinuous, and the (coherent) function $r$ given by $rW = U_A$ has the property that the domain-range restriction of $e$ is uniformly continuous from $A$ to $(M, \{V\colon V \text{ vicinity}, W^{(r)} \subseteq V\})$. In fact, if $(x, y) \in V_A$ then

$$(e(x), e(y)) \in W^{(r)} = \{(f, g) \in M \times M : f \times g(U_A) \subseteq W\}.$$

(c) To show that $e$ is initial, suppose that $D = (T, \delta)$ is a semi-uniform space with $T \subseteq X$ such that the domain-range restriction of $e$ is a morphism $e' : D \to [P, R]$ (i.e., there exists an equicontinuous set $M \subseteq \hom(P, R)$ and a coherent function $r$ [a choice of a vicinity $rW$ in $P$] such that $e' : D \to (M, \{V\colon W^{(r)} \subseteq V\})$ is uniformly continuous). We are going to prove that then $D < A$ for some $A \in \mathscr{A}$. The uniform continuity of $e'$ means that there exists a vicinity $V^* \in \delta$ such that $(e \times e)(V^*) \subseteq W^{(r)}$. Since $rW$ is a vicinity in $P$, it contains some $U_A$ $(A \in \mathscr{A}_0)$, and thus

$$(x, y) \in V^* \text{ implies } (e(x) \times e(y))(U_A) \subseteq W.$$

We are going to prove that $D < A$.

(a) $T \subseteq A$. If contrary, choose $t \in T - A$; further, choose $a \in A$. Since $e(t)(t, t, A) = 0$ and $e(t)(t, a, A) = 1$, we have, for $r = (t, t, A)$ and $s = (t, a, A)$:

$$(e(t)(r), e(t)(s)) \notin W.$$

This is a contradiction since, clearly, $(t, t) \in V^*$ and $(r, s) \in U_A$.

(b) To prove $D < A$, it is sufficient to verify that $V^* \subseteq V_A$. If contrary, choose $(x, y) \in V^* - V_A$ and put $r = (x, x, A)$ and $s = (y, x, A)$. Then $(r, s) \in U_A$ but $(e(x)(r), e(y)(s)) = (0, 1) \notin W$, a contradiction.

This concludes the proof that $e : D \to [P, R]$ is an initial morphism. □

**Example.** **SF** is a proper subcategory of $A^*$**SUnif**; one can prove that the following filter $(X, \mathscr{B})$ is not saturated: Let $X$ be an infinite set and let a subbase of $\mathscr{B}$ be formed by all semi-uniform spaces $(X, \alpha_{\mathscr{F}})$ where $\mathscr{F}$ is a free ultrafilter on $X \times X - \Delta_X$ and

$$\alpha_{\mathscr{F}} = \{U \cup \Delta_X : U = U^{-1} \in \mathscr{F}\}.$$

**Remark.** In [3] we have described the cartesian closed topological hull of the category of uniform spaces as the category of bornological uniform spaces $(X, \alpha, \mathscr{U})$ where $\alpha$ is a uniformity on $X$ and $\mathscr{U}$ is a bornology on $X$ naturally related to each other.

In contrast to bornological uniform spaces, in the concept of saturated filter of semi-uniformities, (1) there is not a single semi-uniformity but various semi-uniformities on various subsets and (2) the bornology (of all underlying sets of the members of the filter) is not related to the structure of semi-uniformities.

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